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Topic 1: Number and Algebra

What do I need to be able to do?

- Apply the hierarchy of operations
- Simplify expressions
- Solve linear equations
- Solve percentage and ratio problems
- Expand brackets
- Expand $(a+b)^n$
- Manipulate surds
- Use the product rule for counting

Key words

- Integer – a number with no fractional part (decimal)
- Product – the answer when values are multiplied together
- Surd – a number that cannot be simplified to remove a square root (or cube root, etc.)
- Rational/irrational – a real number that can/cannot be made by dividing two integers
- Base – the number that gets multiplied when using an exponent
- Binomial – a polynomial with two terms
- Factorial – the product of all integers from the chosen number down to 1. The symbol is !
- Combinations – any of the ways we can combine items, when the order does not matter

Solving percentage and ratio problems

1) x increased by 25% is equal to y decreased by 20%.
What is x as a percentage of y?

$$1.25x = 0.8y$$
$$x = 0.64y \rightarrow 64\%$$

2) $a : b = 2 : 5$. Work out $3a - b : 8b$

$3a - b$ is $3 \times 2 - 5 = 1$ part
$8b$ is $8 \times 5 = 40$ parts
$\Rightarrow 3a - b : 8b = 1 : 40$

The Binomial Expansion

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array}$$

Use Pascal's triangle to find the coefficients of the expansion of $(a+b)^n$

- a starts with the power of n and decreases to 0
- b starts with the power of 0 and increases to n

1) Find the first 3 terms in ascending powers of x in the expansion of $(4-x)^5$

$$(1)(4)^5+(5)(4)^4(-x)^1+(10)(4)^3(-x)^2 = 1024 - 1280x + 640x^2$$

2) Find the coefficient of x^4 in the expansion of $(2+3x)^{10}$

$$x^4 \text{ term} = 210 \times 2^6 \times (3x)^4$$
$$= 210 \times 64 \times 81x^4$$
$$= 1088640x^4 \rightarrow 1088640$$

Manipulating Surds

Multiplying: $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

Dividing: $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$

Adding: $\sqrt{a} + \sqrt{b}$ cannot simplify
$\sqrt{a} + \sqrt{a} = 2\sqrt{a}$

Simplifying: $\sqrt{80} = \sqrt{16 \times 5} = \sqrt{16} \times \sqrt{5} = 4\sqrt{5}$

Rationalising the denominator: $\frac{1}{\sqrt{3}} = \frac{1 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{\sqrt{3}}{3}$

$$\frac{1}{3+\sqrt{5}} = \frac{1(3-\sqrt{5})}{(3+\sqrt{5})(3-\sqrt{5})} = \frac{3-\sqrt{5}}{4}$$

Simplifying expressions

1) $3p^6q \times 5pq^3 = 15p^7q^4$

2) $\frac{20a^4b^4}{16a^3b^9} = \frac{5a}{4b^5}$

Solving linear equations

1) $4(3x+1) = 5x - 10$
$12x + 4 = 5x - 10$
$7x = -14$
$x = -2$

2) $\frac{3a}{5} - \frac{a}{2} = 1$
$6a - 5a = 10$
$a = 10$

Expanding brackets

E.g. Expand and simplify $(4x^2+3)(x^2-4x+5)$

$$= 4x^2(x^2-4x+5) + 3(x^2-4x+5)$$
$$= 4x^4 - 16x^3 + 20x^2 + 3x^2 - 12x + 15$$
$$= 4x^4 - 16x^3 + 23x^2 - 12x + 15$$

The Product Rule for Counting

The number of options for a combination of choices = the product of the number of options for each choice

E.g. How many 5 digit even numbers less than 80000 can be made with the digits 1, 4, 5, 7, and 9 with no repetition of digits?

The last digit must be 4 to generate an even number.
The first digit must be 1, 5 or 7 to remain less than 80000.
We therefore have 3 options for the first digit and 1 for the last. After picking these, there are 3 options left for the second digit, then 3 for the third, then 1 for the fourth.

$$3 \times 3 \times 2 \times 1 \times 1 = 18$$

Topic 2: Algebra II

What do I need to be able to do?

- Factorise expressions by taking out common factors
- Factorise quadratic expressions
- Factorise the difference of two squares
- Change the subject of a formula
- Simplify combinations of algebraic fractions
- Solve linear equations containing algebraic fractions
- Complete the square

Key words

- Equation – a mathematical statement containing an equals sign. It will have a finite set of solutions
- Formula – a relationship between variables representing real-life quantities
- Coefficient – a number multiplied by a variable
- Constant – a fixed value: a term containing no variable

Algebraic fractions

Algebraic fractions behave as numerical fractions.

When simplifying algebraic fractions, where possible factorise the numerator and denominator and then cancel out common factors.

1) $\frac{2x^2+11x+12}{x^2+7x+12} = \frac{(2x+3)(x+4)}{(x+3)(x+4)} = \frac{(2x+3)\cancel{(x+4)}}{(x+3)\cancel{(x+4)}} = \frac{2x+3}{x+3}$

We can 'cross cancel' when multiplying fractions; that is, we can cancel a common factor from the numerator of one fraction and denominator of another.

2) $\frac{x}{3q} \times \frac{5q}{4y} = \frac{x}{3\cancel{q}} \times \frac{5\cancel{q}}{4y} = \frac{x}{3} \times \frac{5}{4y} = \frac{5x}{12y}$

Solving equations containing fractions

Find the LCM of the denominators. Clear the fractions by multiplying through by this LCM.

E.g. $\frac{x}{3} + \frac{x+2}{5} = -2$

The LCM of 3 and 5 is 15:

$$\frac{x}{3} \times 15 + \frac{x+2}{5} \times 15 = -2 \times 15$$
$$5x + 3(x+2) = -30$$
$$5x + 3x + 6 = -30$$
$$8x = -36$$
$$x = -4.5$$

The difference of two squares

$a^2 - b^2 = (a+b)(a-b)$ E.g. $a^2b^2 - 81t^2 = (ab+9t)(ab-9t)$

Completing the square

Given that a quadratic can be expressed in the form

$$p(x+q)^2 + r$$

expand this and compare coefficients to determine p, q and r

E.g. $2x^2 - 16x + 22 = p(x+q)^2 + r$

Expand $2x^2 - 16x + 22 = px^2 + 2pqx + pq^2 + r$

Equate coefficients of x^2 $2 = p$

Equate coefficients of x
$$-16 = 2pq$$
$$-16 = 4q$$
$$-4 = q$$

Equate constants
$$22 = pq^2 + r$$
$$22 = 32 + r$$
$$-10 = r$$

So $2x^2 - 16x + 22 = 2(x-4)^2 - 10$

Factorising by grouping

Take out a common factor of pairs of terms

E.g. $xp + xq + yp + yq = x(p+q) + y(p+q)$
$= (p+q)(x+y)$

Factorising quadratics

Quadratics in *one* variable: Multiply the coefficient of x^2 by the constant. Find the pair of numbers that have a product of this number and that sum to the coefficient of x. Split the middle term using this pair, then factorise by grouping.

1) $6x^2 - 7x - 3$ → $6 \times -3 = -18$
$= 6x^2 + 2x - 9x - 3$
$= 2x(3x+1) - 3(3x+1)$
$= (3x+1)(2x-3)$

Quadratics in *two* variables: Follow the same process, looking only at the coefficients.

2) $2x^2 - 9xy + 10y^2$ → $2 \times 10 = 20$
$= 2x^2 - 5xy - 4xy + 10y^2$
$= x(2x-5y) - 2y(2x-5y)$
$= (2x-5y)(x-2y)$

Topic 3: Algebra III

What do I need to be able to do?

- State the domain and range of a function
- Find composite functions
- Sketch the graph of a function
- Find the equation of a line
- Recognise the shape of quadratic and exponential functions
- Find inverse functions
- Interpret piecewise functions

Key words

- Function – a mapping that has a unique output for every input
- Domain – the set of input values of a function
- Range – the set of output values of a function
- Composite function – when multiple functions act in succession
- Inverse function – the function that 'undoes' the effect of the original function
- Piecewise – a function with multiple parts to its domain

Domain and range

The domain is the set of inputs of a function, which is often stated.

E.g. $f(x) = 2x + 3 \qquad 1 \leq x < 4$
The domain is stated $1 \leq x < 4$

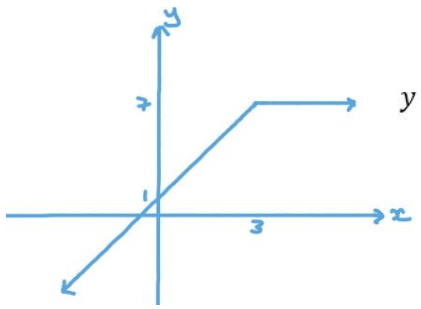
The range is the set of outputs of a function, which is often found by sketching $f(x)$ or substituting in the domain critical values.

In the above example, the range is $5 \leq f(x) < 11$

Piecewise functions

You should be able to sketch graphs of functions with up to 3 parts to their domains.

$$f(x) = \begin{cases} 2x + 1 & x \leq 3 \\ 7 & x > 3 \end{cases}$$



$y = f(x)$

Inverse functions

Written $f^{-1}(x)$.

The graphs of $f(x)$ and $f^{-1}(x)$ are reflections of each other in the line $y = x$.

The domain of $f(x)$ is the range of $f^{-1}(x)$.
The range of $f(x)$ is the domain of $f^{-1}(x)$.

To find the inverse of a function:
- Interchange x and y
- Make y the subject

Composite functions

Always apply the inside function first.

To find $fg(x)$ do $g(x)$ first then substitute your answer into $f(x)$ to find the answer.

E.g. $f(x) = x^2$ and $g(x) = x + 1$

a) Find $fg(2)$:

$$g(2) = 2 + 1 = 3$$
$$f(3) = 3^2 = 9$$

b) Find $gf(x)$:

$$f(x) = x^2$$
$$g(x^2) = x^2 + 1$$

Quadratic functions

The line of symmetry passes through the vertex.

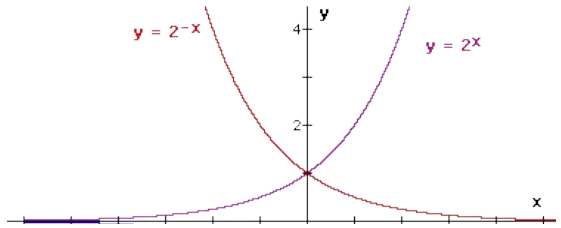
The vertex can be found by completing the square: $p(x + q)^2 + r$

→ vertex at $(-q,\ r)$

Sketching functions

A sketch requires:
- The correct shape for the function in each quadrant
- Labelling of axes intercepts
- Labelling of any vertices

Exponential functions



Topic 4: Algebra IV

What do I need to be able to do?

- Solve quadratic equations
- Solve simultaneous equations, including in three unknowns
- Solve linear and quadratic inequalities
- Divide a polynomial by a linear factor
- Use the factor theorem
- Construct mathematical proof using algebra
- Find and use the nth term of sequences
- Find the limit of a sequence

Key words

- Inequality – a mathematical statement comparing two values
- Polynomial – an expression that can have constants, variables and exponents that can be combined using addition, subtraction, multiplication and division but
  - No division by a variable
  - Exponents can only be 0 or a positive integer
- Proof – logical mathematical arguments used to show the truth of a statement

The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic inequalities

- Solve $ax^2 + bx + c = 0$ to get the critical values p and q
- If $a < 0$ then multiply throughout by -1
- If the inequality is $<$ the solution is $p < x < q$
- If the inequality is $>$ the solution is $x < p$ or $x > q$

The factor theorem

If $f(a) = 0$, then $(x - a)$ is a factor of $f(x)$.

If $(x - a)$ is a factor of $f(x)$, then $f(a) = 0$.

This theorem allows us to divide a cubic function by a linear factor, in order to fully factorise the cubic function.

Sequences

To find the limiting value of a sequence, consider the nth term as $n \to \infty$.

E.g. $\frac{5n-4}{2n+1}$

Divide throughout by n: $\frac{\frac{5n}{n}-\frac{4}{n}}{\frac{2n}{n}+\frac{1}{n}} = \frac{5-\frac{4}{n}}{2+\frac{1}{n}}$

As n approaches infinity $\frac{1}{n}$ and $\frac{4}{n}$ both approach zero.

So $\frac{5-\frac{4}{n}}{2+\frac{1}{n}} \to \frac{5-0}{2+0} = \frac{5}{2}$

Disguised quadratics

Some equations can be transformed into quadratic equations to be solved. We must check that any solutions are not extraneous.

1) $x^4 - 7x^2 + 6 = 0$

Let $y = x^2$, then $y^2 = x^4$

$$y^2 - 7y + 6 = 0$$
$$(y - 6)(y - 1) = 0$$
$$y = 6 \text{ or } y = 1$$
$$x = \sqrt{6} \text{ or } x = \sqrt{1} = \pm 1$$

$x = -1$ is extraneous

2) $4^x + 2^x = 0$

Let $y = 2^x$, then $y^2 = (2^x)^2 = 2^{2x} = 4^x$

$$y^2 + y = 0$$
$$y(y + 1) = 0$$
$$y = 0 \text{ or } y = -1$$
$$x = 0$$

Simultaneous equations in 3 unknowns

- Eliminate one unknown by combining a pair of equations
- Eliminate the same unknown by combining another pair of equations
- Solve the resulting pair of equations using methods for simultaneous equations in 2 unknowns
- Substitute the two found unknowns into one of the original equations to find the third unknown

Proof

In a mathematical proof you must:

- State any information or assumptions you are using
- Show every step clearly
- Each step should follow logically from the previous step
- Make sure you have covered all possible cases
- Write a statement of proof at the end of your working

Topic 5: Coordinate Geometry

What do I need to be able to do?

- Know and use the rules for parallel and perpendicular lines
- Calculate the distance between two points
- Find the midpoint of a line segment
- Find the point of intersection of two lines
- Divide a line segment in a given ratio
- Find the equation of a circle
- Solve problems involving tangents to circles

Key words

- Gradient – how steep a line is
- Parallel – always the same distance apart and never touching
- Perpendicular – at right angles
- Line segment – a finite part of a line
- Tangent – a line that touches a curve at only one point

The equation of a straight line

There are several ways you can write an equation of a straight line:

Form	When it's used
	You have the gradient and y-intercept
	You have the gradient and a point on the line
	When we want integer values

Equation of a circle

$$(x-a)^2+(y-b)^2=r^2$$

Is the equation of circle centre (a,b) and radius r.

You may be given the equation in the form:

$$x^2+y^2-2ax-2by+a^2+b^2=r^2=0$$

In this case you need to complete the square for the x and y terms to find the centre and radius.

E.g. $x^2+y^2-14x+16y-12=0$

Rearrange $x^2-14x+y^2+16y-12=0$

Complete the square $(x-7)^2-7^2+(y+8)^2-8^2-12=0$

Collect constants $(x-7)^2+(y+8)^2=125$

So the centre is $(7,-8)$ and the radius is $\sqrt{125}$

Parallel and perpendicular

Parallel lines: same gradient.

Perpendicular lines: the product of the gradients is -1. The gradients are the negative reciprocals of each other.

Solving circle problems

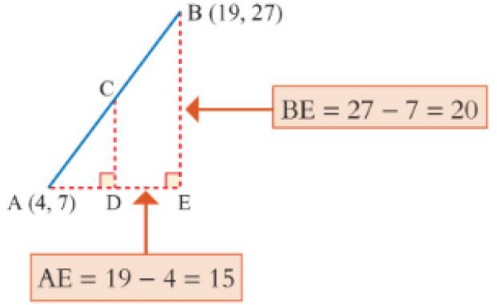
You need to know the following circle theorems, to use them in conjunction with the equation of a circle.

- The angle in a semi-circle is $90°$
- The perpendicular from the centre to a chord bisects the chord
- The angle between a tangent and radius is $90°$
- Two tangents from a point to a circle are equal in length

Dividing a line in a given ratio

Use the idea of similar triangles to calculate the point.

E.g.



The point C divides the line AB in the ratio 2:3.

So $AD=\frac{2}{5}\times 15=6$ and $CD=\frac{2}{5}\times 20=8$.

Point C is therefore 6 along and 8 up from $A(4,7)$:

$$(4+6,7+8)=(10,15)$$

Intersections

To find the point of intersection of two lines, solve as simultaneous equations.

Distance between two points

$$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

Gradient

$$m=\frac{y_2-y_1}{x_2-x_1}$$

Midpoint of a line segment

$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$$

Topic 6: Geometry I

What do I need to be able to do?

- Apply circle theorems
- Construct geometric proof
- Calculate the sine, cosine and tangent of any angle
- Know the exact trigonometric ratios for $30°$, $45°$ and $60°$
- Sketch and recognise the trigonometric graphs
- Solve trigonometric equations
- Know and use the trigonometric identities

Key words

- Identity – an equation that is always true, no matter what values are substituted

Circle theorems

- Angle at the centre is double the angle at the circumference
- The angle in a semi-circle is $90°$
- Angles in the same segment are equal
- Opposite angles of a cyclic quadrilateral add up to $180°$
- Alternate segment theorem

Geometric proof

In a geometric proof you must:

- State any geometric properties you are using
- Use the correct notation for angles, shapes and line segments
- Show every step clearly
- Each step should follow logically from the previous step
- Write a statement of proof at the end of your working

Exact trigonometric values

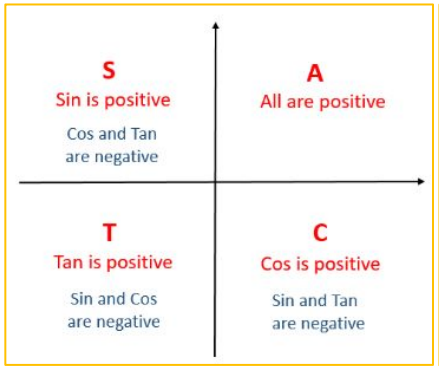
sin			
cos			
tan		1	

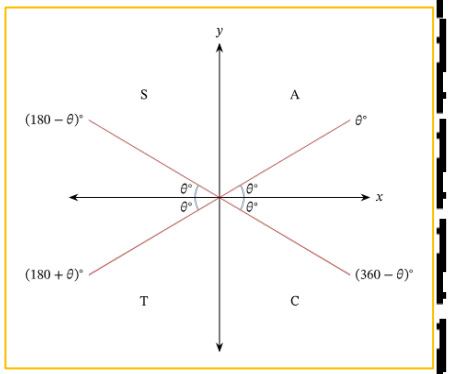
Solving trigonometric equations

Technique	Example
Rearrange to make sin, cos or tan the subject	
Factorise if possible	
If it is a mixture of sin and cos, use identity 2	
If it is a mixture of sin and cos in a quadratic use identity 1	

Finding sin, cos and tan fo any angle

You can use the graphs of the trigonometric ratios or the CAST diagram to find sin, cos or tan of any positive or negative angle, using the corresponding acute angle made with the x-axis.





E.g. Solve $sinx = 0.3$ in the interval $0 \leq x \leq 360$

$$x = sin^{-1}(0.3)$$

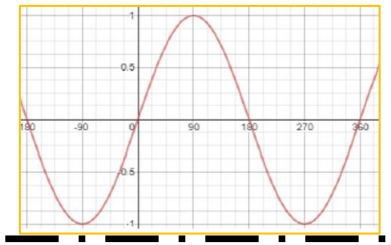
Your calculator will give you the *principal value:* $x = 17°$
Sin is also positive in the second quadrant: $x = 180 - 17 = 163°$

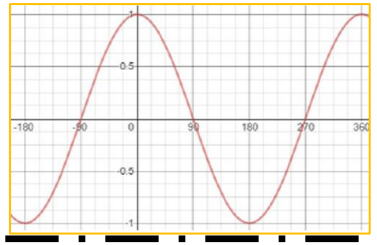
Trigonometric identities

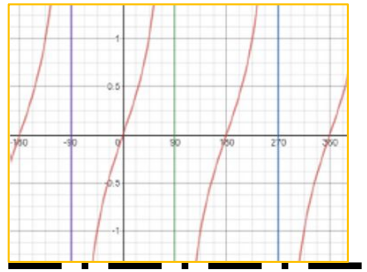
$$sin^2\theta + cos^2\theta \equiv 1 \quad (1)$$

$$tan\theta \equiv \frac{sin\theta}{cos\theta} \quad (2)$$

Graphs of sin/cos/tan





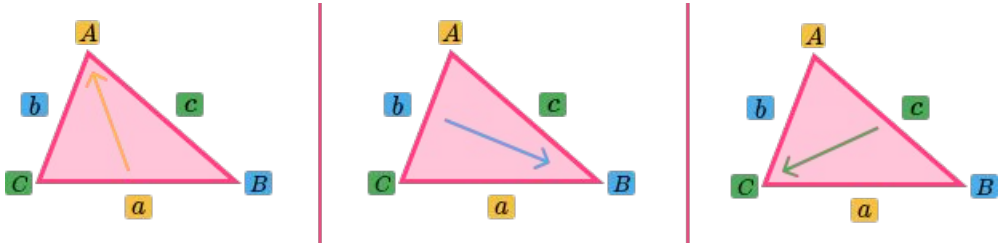


Topic 7: Geometry II

What do I need to be able to do?

- Use the sine rule to find a missing side or angle
- Use the cosine rule to find a missing side or angle
- Find the area of a triangle using an appropriate formula
- Solve problems involving triangles
- Solve problems involving lines and planes in 3D

The sine rule



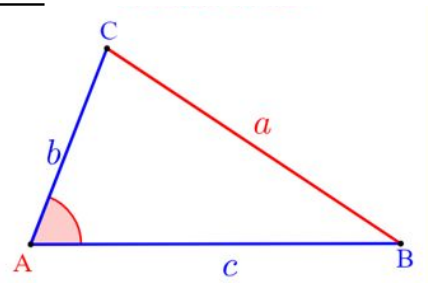
To find a missing side: $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}$

To find a missing angle: $\frac{sinA}{a} = \frac{sinB}{b} = \frac{sinC}{c}$

Use the sine rule when you have opposite pairs of angles and sides.

The sine rule sometimes produces two possible solutions for a missing angle.
Use $sin\theta = \sin(180 - \theta)$ to find the other.

The cosine rule



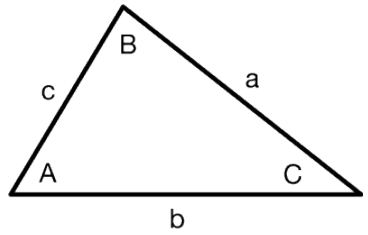
To find a missing side: $a^2 = b^2 + c^2 - 2bccosA$

To find a missing angle: $cosA = \frac{b^2+c^2-a^2}{2bc}$

Use the cosine rule when you either:

- know two sides and the angle between them, and want to find the third side
- Know three sides, and want to find the angle

The area of a triangle



$$Area = \frac{1}{2}absinC$$

Topic 8: Calculus

What do I need to be able to do?

- Differentiate positive and negative powers of a variable
- Differentiate sums and differences of functions of x
- Use differentiation to work out the gradient of a curve
- Identify stationary points on a polynomial curve
- Find the equation of a tangent or a normal to a curve
- Identify when a function is increasing/decreasing
- Use the second derivative to determine the nature of stationary points

Key words

- Derivative – rate of change of a function, equal to the instantaneous gradient. Denoted by $\frac{dy}{dx}$ or $f'(x)$.
- Differentiate – to find the derivative
- Tangent – a line that touches a curve at one point, with equal gradient
- Normal – a line perpendicular to a tangent
- Stationary point – a point on a curve where the gradient is 0

Differentiating

If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$

If $y = c$ then $\frac{dy}{dx} = 0$

Multiply by the power and decrease the power by 1.

1) $y = 4x^5$ → $\frac{dy}{dx} = 20x^4$

2) $y = 3x^{\frac{1}{3}}$ → $\frac{dy}{dx} = x^{-\frac{2}{3}}$

3) $y = \frac{5}{x^2}$ → first rearrange into the form ax^n: $5x^{-2}$

$$\frac{dy}{dx} = -10x^{-3}$$

Finding the gradient

Differentiating finds the *gradient function* or *rate of change* of a function.

We can substitute in an x-coordinate to find the gradient at that point.

E.g. $y = x^4$ → $\frac{dy}{dx} = 4x^3$

At the point $(2, 8)$: $\frac{dy}{dx} = 4 \times 2^3 = 32$

The second derivative

Found by differentiating $\frac{dy}{dx}$, notation $\frac{d^2y}{dx^2}$

This finds the rate of change of $\frac{dy}{dx}$

Increasing/decreasing functions

Increasing: $\frac{dy}{dx} > 0$ Decreasing: $\frac{dy}{dx} < 0$

We can find the part of a domain in which a function is increasing/decreasing.

E.g. $y = 8x - x^2$ → $\frac{dy}{dx} = 8 - 2x$

Increasing when...	Decreasing when...
$8 - 2x > 0$	$8 - 2x < 0$
$8 > 2x$	$8 < 2x$
$x < 4$	$x > 4$

Tangents and Normals

The gradient of a tangent, m_1, is the value of $\frac{dy}{dx}$ at the point.

The gradient of a normal, m_2, is $-\frac{1}{m_1}$.

E.g. Equation of the normal to $y = 5x^2 - 20x$ at the point $(1, -15)$:

$\frac{dy}{dx} = 10x - 20$ → gradient $= 10 \times 1 - 20 = -10$

Therefore the gradient of the normal is $\frac{1}{10}$

Equation of the normal: $y + 15 = \frac{1}{10}(x - 1)$

Stationary points

Solving $\frac{dy}{dx} = 0$ gives the x-coordinate of the stationary points on a curve.

Substituting into $\frac{d^2y}{dx^2}$ gives the nature of the stationary point: > 0 is a minimum, < 0 is a maximum point.

E.g. $y = x^3 - 12x$ has derivative $\frac{dy}{dx} = 3x^2 - 12$

$$3x^2 - 12 = 0$$
$$x = \pm 2$$

At $x = 2$: $\frac{d^2y}{dx^2} = 6x = 6 \times 2 = 12$ (minimum)

At $x = -2$: $\frac{d^2y}{dx^2} = 6x = 6 \times -2 = -12$ (maximum)

Topic 9: Matrices

What do I need to be able to do?

- Multiply a 2 x 2 matrix by a scalar
- Multiply and 2 x 2 matrix by a 2 x 1 or 2 x 2 matrix
- Recognise the identity matrix
- Recognise and use transformation matrices
- Apply a combination of transformations
- Find the matrix representing a combination of transformations

Key words

- Matrix – an arrangement of information presented in columns and rows
- Element – each of the numbers in a matrix
- Scalar – a quantity having only magnitude, not direction
- Unit square – the square with vertices $(0,0)$, $(0,1)$, $(1,1)$ and $(1,0)$

Multiplying matrices

- By a scalar: multiply each element by the scalar

E.g. $$5\begin{bmatrix}0 & 1\\-4 & 3\end{bmatrix}=\begin{bmatrix}0 & 5\\-20 & 15\end{bmatrix}$$

- 2×2 by 2×1: multiply each row of the 2×2 by the column of the 2×1 matrix

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}p\\q\end{bmatrix}=\begin{bmatrix}ap+bq\\cp+dq\end{bmatrix}$$

- 2×2 by 2×2: multiply each row of the first by each column of the second matrix

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}j & k\\l & m\end{bmatrix}=\begin{bmatrix}aj+bl & ak+bm\\cj+dl & ck+dm\end{bmatrix}$$

Non-commutativity

Matrix multiplication is non-commutative. This means the order in which we multiply matters, and $\boldsymbol{AB\neq BA}$.

The identity matrix

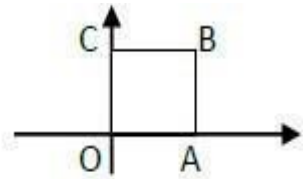
$$\boldsymbol{I}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

When multiplying by the $\boldsymbol{I}$, no transformation occurs.

$\boldsymbol{MI=IM=M}$ for any matrix **M**.

Transformations of the unit square

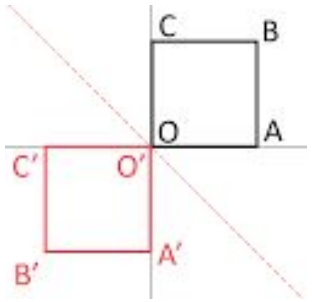
The unit square has vertices $O(0,0)$, $A(1,0)$, $B(1,1)$ and $C(0,1)$:



To find the matrix representing a particular transformation, consider the images of vertices A and C.

E.g. Find the matrix representing a reflection in the line $y=-x$

Sketch:



A has been mapped to $A'\begin{pmatrix}0\\-1\end{pmatrix}$ and C has been mapped to $C'\begin{pmatrix}-1\\0\end{pmatrix}$

Write these two position vectors side by side: $\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$

Combining transformations

A transformation represented by matrix $\boldsymbol{P}$ can be followed by a transformation represented by matrix $\boldsymbol{Q}$:

The combined transformation is $\boldsymbol{QP}$.

E.g. $\mathbf{P}=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ and $\boldsymbol{Q}=\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$

$$\boldsymbol{QP}=\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$$

This is the transformation matrix that maps vertex A to $(1,0)$ and vertex C to $(0,-1)$.

This is a reflection in the x-axis.

Alternative method

We can determine what transformation is represented by $\boldsymbol{P}$ and $\boldsymbol{Q}$ individually, apply these to the unit square in turn, then write the final position vectors as a matrix.

Required transformations

- Reflection in: x-axis, y-axis, $y=x$, $y=-x$
- Rotations about the origin through: $90°$, $180°$, $270°$
- Enlargements, centre origin